

THE FRACTIONAL HARTREE EQUATION WITHOUT THE AMBROSETTI-RABINOWITZ CONDITION

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ABSTRACT. We consider a class of pseudo-relativistic Hartree equations in presence of general nonlinearities not satisfying the Ambrosetti-Rabinowitz condition. Using variational methods based on critical point theory, we show the existence of two non trivial signed solutions, one positive and one negative.

1. INTRODUCTION

In this paper we deal with a general class of pseudo-relativistic Schrödinger equations with a Hartree non linearity. Such equations emerge from the description of pseudorelativistic boson stars (see [16] for a physical derivation of the problem), but also as the mean field limit description of a quantum relativistic Bose gas (see [9] and [17]). Fröhlich and Lenzmann in [11] and [10] approached the problems of existence, blowing up and stability of solutions. The problem they studied in [11] took the following form:

$$(1.1) \quad i\psi_t = \sqrt{-\Delta + m^2}\psi - \left(\frac{1}{|x|} * |\psi|^2\right) \psi \text{ in } \mathbb{R}^3,$$

Here ψ is a complex valued wave function which describes the quantum status of a particle, while the operator involving the square root represents its relativistic kinetic and rest energies, and reduces to the usual half Laplacian $(-\Delta)^{1/2}$ when $m = 0$. Besides, the term $1/|x|$ inside the convolution product stands for the Newtonian gravitational potential in \mathbb{R}^3 and represents repulsive forces among the particles.

In [24], a generalized version of (1.1) is studied, allowing for an additional potential term $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, which takes into account other external forces, and, in addition, a general field potential W replaces the Newtonian one. In this setting, equation (1.1) takes the following form:

$$(1.2) \quad i\psi_t = \sqrt{-\Delta + m^2}\psi - \lambda (W * |\psi|^2) \psi - f(x, \psi) \text{ in } \mathbb{R}^N,$$

with $\lambda \in \mathbb{R}$.

In this paper we search solutions of a problem corresponding to (1.2) but settled in a bounded domain Ω of \mathbb{R}^N . This allows us to remove the hypothesis of radial symmetry of the solutions and of the potential f assumed in [24]. The problem we

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study is the following one:

$$(1.3) \quad \begin{cases} i\psi_t(x, t) = \sqrt{-\Delta + m^2}\psi(x, t) - \\ \quad -\lambda \left(\int_{\Omega} G(x, y)|\psi(y, t)|^2 dy \right) \psi(x, t) - f(x, \psi(x, t)) & \text{in } \Omega, \\ \psi(x, t) = 0 & \text{on } \partial\Omega, \quad \forall t \end{cases}$$

with $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^N$ bounded.

In (1.3), passing from \mathbb{R}^N to Ω , we have replaced the Newtonian-like kernel $W(x - y)$ with the Green function $G(x, y)$ associated to the Laplace operator in Ω (indeed this is the Coulomb-type interaction between particles in boson stars), and we consider homogeneous boundary conditions on $\partial\Omega$. In this way, the corresponding potential ϕ at time t takes the form $\phi(x) = \int_{\Omega} G(x, y)|\psi(y, t)|^2 dy$. From

now on we will adopt the symbol $\langle G, \psi \rangle = \int_{\Omega} G(x, y)|\psi(y, t)|^2 dy$ for the previous potential term. We also note that potential $\phi(x) = \langle G, \psi \rangle$ is the solution of the linear problem

$$(1.4) \quad \begin{cases} -\Delta\phi(x, t) = 4\pi|\psi(x, t)|^2 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

for every t , so that problem (1.3) can be written as a system with an additional equation for ϕ , as similarly done in [2], [8], [7], [22].

It is worth reminding some general properties of Green functions for C^1 bounded domains Ω , which we shall use later (for instance, see [13]). Green functions $G : \Omega \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ are non negative, symmetric with respect to their variables, and when $N \geq 3$ they verify the inequality $G(x, y) \leq C|x - y|^{2-N}$, where the right hand side is the kernel of the Newtonian like potential. More generally, inspired by the above inequality, in our setting we consider a function G which is symmetric, non negative and satisfies certain integrability conditions. To be precise, we will require $G(x, y) \leq W(x - y)$, with W satisfying some integrability conditions which cover the case of the Newtonian kernel for $N \geq 3$.

In order to obtain existence of solutions for problem (1.3), it is crucial to specify some hypothesis on the external potential $F(x, \psi) = \int_0^\psi f(x, s)ds$. The prototype for F is a power-like potential, so it is natural to require $F(x, s) = F(x, |s|)$ and $f(x, e^{i\theta}|s|) = e^{i\theta}f(x, |s|)$. This is not restrictive in the setting of Abelian Gauge Theories (see [3], [23]), and it allows us to search for real solutions of the stationary equation associated to (1.3). Indeed, we focus on solutions in the form of solitary waves, i.e. on functions of the form

$$(1.5) \quad \psi(x, t) = e^{-i\omega t}u(x),$$

where $\omega \in \mathbb{R}$ and $u : \Omega \rightarrow \mathbb{R}$.

After substitution of (1.5) into (1.3), and considering that the operator $\sqrt{-\Delta + m^2}$ acts only on the spatial coordinates, we see that function u satisfies the following stationary equation:

$$(1.6) \quad \begin{cases} \sqrt{-\Delta + m^2}u - \omega u - \lambda \langle G, u^2 \rangle u - f(x, u) = 0 & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We remark that a standard assumption on F for solving stationary equations like (1.6) is the fulfillment of the usual Ambrosetti-Rabinowitz condition, see [3], [23], [25] (or a reversed one (see [24])). We recall that this condition reads as follows: there exists $\mu > 2$ such that

$$(1.7) \quad 0 < \mu F(x, s) \leq s f(x, s) \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

Condition (1.7) is very useful to prove that Palais–Smale sequences are bounded, and in turn that Palais Smale condition (PS-condition in short) holds, so that an essential ingredient in variational methods is guaranteed.

Although very convenient, condition (1.7) rules out many interesting non linearities. For this reason, many efforts have been done recently to remove or relax it (see Li, Wang, Zeng [14], [15], [18], [19], Myagaki and Souto [21]). In this paper, we adopt the strategy of Mugnai and Papageorgiou ([26]), which consists in requiring a quasi-monotone property for the function

$$(1.8) \quad \sigma(x, s) = f(x, s)s - 2F(x, s) \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R},$$

see Section 2 for the precise assumptions. The purpose of this paper is to show that, under our weak assumptions, equation (1.6) has two non trivial bounded solutions. In order to prove this result, we will employ an *a priori* estimate for solutions of (1.6) of independent interest, see Proposition 1 below.

2. EXTENDED PROBLEM AND ASSUMPTIONS

We start this section by reviewing the essential tools to face problem (1.6). We follow the idea of extending equation (1.6) to an equivalent one in higher dimension, by means of the Dirichlet-to-Neumann operator $-\frac{\partial}{\partial x_{N+1}} \Big|_{x_{N+1}=0}$ (see [4] for this procedure in bounded domains and [5] for the whole spatial domain). This method leads us to consider the following problem:

$$(2.1) \quad \begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathcal{C}, \\ -\frac{\partial v}{\partial x_{N+1}} = \omega v + \lambda \langle G, v^2 \rangle v + f(x, v) & \text{on } \Omega \times \{0\}, \\ v = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times [0, \infty). \end{cases}$$

where $\mathcal{C} = \Omega \times (0, \infty)$ is the positive half cylinder with base Ω and $\partial_L \mathcal{C}$ is its lateral boundary. As in [4], we have that if v satisfies (2.1), then its trace $u(\cdot) := v(\cdot, 0)$ on $\Omega \times \{0\}$ satisfies problem (1.6).

In order to define a weak solution of (2.1) (see Cabré and Tan [4]), we introduce the Sobolev space

$$(2.2) \quad H_{0,L}^1(\mathcal{C}) = \left\{ v \in H^1(\mathcal{C}) : v = 0 \text{ a.e. on } \partial_L \mathcal{C} \right\},$$

equipped with the inner product

$$\langle u, v \rangle = \int_{\mathcal{C}} (Du \cdot Dv + m^2 uv) dx dx_{N+1},$$

which makes $H_{0,L}^1(\mathcal{C})$ a Hilbert space with respect to the induced norm (here and henceforth we shall denote by x a general point of Ω and by x_{N+1} an element of $[0, \infty)$).

Hence, we can give the following

Definition 1 (Weak Solution). *A function $v \in H_{0,L}^1(\mathcal{C})$ is a weak solution of problem (2.1) if*

$$(2.3) \quad \int_{\mathcal{C}} [Dv \cdot Dw + m^2 vw] dx dx_{N+1} = \int_{\Omega} [\omega v + \lambda \langle G, v^2 \rangle v + f(x, v)] w dx$$

for every $w \in H_{0,L}^1(\mathcal{C})$.

Recalling that G is symmetric, it is easy to see that problem (2.3) is of variational nature, so that a function v in $H_{0,L}^1(\mathcal{C})$ satisfies (2.3) if and only if it is a critical point of the following energy functional $J : H_{0,L}^1(\mathcal{C}) \rightarrow \mathbb{R}$ defined as

$$(2.4) \quad J(v) = \frac{1}{2} \int_{\mathcal{C}} [|Dv|^2 + m^2 v^2] dx dx_{N+1} - \int_{\Omega} \left[\frac{\omega}{2} v^2 + \frac{\lambda}{4} \langle G, v^2 \rangle v^2 + F(x, v) \right] dx,$$

or, in compact form,

$$(2.5) \quad J(v) = \frac{1}{2} \|Dv\|_2^2 + \frac{m^2}{2} \|v\|_2^2 - \frac{\omega}{2} \|v\|_2^2 - \frac{\lambda}{4} \int_{\Omega} \langle G, v^2 \rangle v^2 dx - \int_{\Omega} F(x, v) dx,$$

where $|\cdot|_p$ and $\|\cdot\|_p$ denote the L^p norm in Ω and in \mathcal{C} , respectively. The derivative of functional J acts on any function $w \in H_{0,L}^1(\mathcal{C})$ in the following way:

$$(2.6) \quad \begin{aligned} J'(v)w &= \langle Dv, Dw \rangle_{2,N+1} + m^2 \langle v, w \rangle_{2,N+1} - \omega \langle v, w \rangle_{2,N} \\ &\quad - \lambda \int_{\Omega} \langle G, v^2 \rangle vw dx - \int_{\Omega} f(x, v)w dx, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{p,n}$ denotes the duality product in $[L^p(\mathcal{C})]' \times L^p(\mathcal{C})$ when $n = N + 1$ or in $[L^p(\Omega)]' \times L^p(\Omega)$ when $n = N$.

Of course, the previous considerations are just formal ones, if we don't assume appropriate conditions on f and G . For this, throughout this paper we make the following assumptions, which guarantee that the formal considerations above are indeed true:

$\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary of class $C^{2,\alpha}$;

(H): $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, with $f(x, 0) = 0$ for a.e. $x \in \Omega$.

Moreover, if $F(x, s) := \int_0^s f(x, \tau) d\tau$, we suppose that

Hi): there exist $c > 0$, $a \in L^\infty(\Omega)$, $a \geq 0$ a.e. in Ω , and $r \in \left(2, \frac{2N}{N-1}\right)$ such that

$$|f(x, s)| \leq a(x) + c|s|^{r-1}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$;

Hii): $\lim_{|s| \rightarrow \infty} \frac{F(x, s)}{s^2} = +\infty$ uniformly for a.e. $x \in \Omega$;

Hiii): if $\sigma(x, s) := f(x, s)s - 2F(x, s)$, there exists $\beta^* \in L_+^1(\Omega)$ s.t.

$$\sigma(x, s) \leq \sigma(x, t) + \beta^*(x)$$

for a.e. $x \in \Omega$ and all $0 \leq s \leq t$ or $t \leq s \leq 0$;

Hiv): there exists $\theta \in L_+^\infty(\Omega)$ with $\theta_\infty = |\theta|_\infty < m - \omega$ such that

$$\limsup_{s \rightarrow 0} \frac{F(x, s)}{s^2} \leq \frac{\theta(x)}{2}.$$

Concerning G , as a generalization of the Green function of the domain Ω , which belongs to $L^r(\Omega)$ for $r < N$, we assume the natural hypothesis

(**G**): $G \geq 0$, $G(x, y) = G(y, x)$ and $G(x, y) \leq W(x - y)$ for every $(x, y) \in \Omega \times \Omega$, where $W \geq 0$ in \mathbb{R}^N , $W \in L^r(\mathbb{R}^N)$ for some $r \in (\frac{N}{2}, \infty)$ and $W = 0$ in $\mathbb{R}^N \setminus \Omega$.

3. TECHNICAL INEQUALITIES

We now establish some useful inequalities which will be used extensively throughout the paper.

We start with the continuous inclusions (see [4, Lemma 2.4])

$$(3.1) \quad H_{0,L}^1(\mathcal{C}) \hookrightarrow L^r(\Omega) \quad \text{for all } r \in \left[1, \frac{2N}{N-1}\right],$$

and the compact ones (see [4, Lemma 2.5])

$$(3.2) \quad H_{0,L}^1(\mathcal{C}) \hookrightarrow L^r(\Omega) \quad \text{for all } r \in \left[1, \frac{2N}{N-1}\right).$$

Now, take $v \in \mathcal{C}^\infty(\mathbb{R}_+^{N+1}) \cap H_{0,L}^1(\mathcal{C})$; then, proceeding as in [24],

$$(3.3) \quad \begin{aligned} \int_{\Omega} |v(x, 0)|^q dx &= \int_{\Omega} \left(- \int_0^\infty \frac{\partial}{\partial x_{N+1}} |v(x, x_{N+1})|^q dx_{N+1} \right) dx \\ &= -q \int_{\mathcal{C}} |v(x, x_{N+1})|^{q-2} v(x, x_{N+1}) \frac{\partial v}{\partial x_{N+1}}(x, x_{N+1}) dx \, dx_{N+1} \end{aligned}$$

Applying the Hölder inequality with exponent 2, we get

$$(3.4) \quad \int_{\Omega} |v(x, 0)|^q dx \leq q \|v^{q-1}\|_2 \left\| \frac{\partial v}{\partial x_{N+1}} \right\|_2 \leq q \|v\|_{2(q-1)}^{q-1} \|Dv\|_2.$$

By interpolation for $2(q-1)$ between 2 and $2^\sharp = \frac{2N}{N-1}$, followed by the Sobolev embedding inequality, we find the trace inequality

$$|v|_q \leq S_q \|v\|$$

for every $v \in \mathcal{C}^\infty(\mathbb{R}_+^{N+1}) \cap H_{0,L}^1(\mathcal{C})$, where S_q is an absolute positive constant.

Moreover, if we use the Cauchy inequality in (3.3), when $q = 2$ we obtain

$$(3.5) \quad |v|_2^2 = \int_{\Omega} |v(x, 0)|^2 dx \leq \epsilon \|v\|_2^2 + \frac{1}{\epsilon} \left\| \frac{\partial v}{\partial x_{N+1}} \right\|_2^2.$$

In particular, choosing $\epsilon = m$, (3.5) gives the following estimate for the trace norm:

$$(3.6) \quad |v|_2^2 \leq m \|v\|_2^2 + \frac{1}{m} \left\| \frac{\partial v}{\partial x_{N+1}} \right\|_2^2 \leq m \|v\|_2^2 + \frac{1}{m} \|Dv\|_2^2 \quad \forall v \in \mathcal{C}^\infty(\mathbb{R}_+^{N+1}) \cap H_{0,L}^1(\mathcal{C}).$$

Finally, by density, we have that all the inequalities above hold for every $v \in H_{0,L}^1(\mathcal{C})$.

Now, we proceed by inferring some inequalities on F which come directly from hypothesis (**H**). First, a direct integration of **Hi**) gives

$$(3.7) \quad |F(x, s)| \leq a(x)|s| + \frac{c}{r}|s|^r \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

Furthermore, from **Hiv**) we can say that for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$(3.8) \quad F(x, s) \leq \frac{\theta(x) + \epsilon}{2} s^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } |s| < \delta.$$

From (3.7) and (3.8), we deduce that

$$(3.9) \quad F(x, s) \leq \frac{\theta(x) + \epsilon}{2} s^2 + C_\epsilon |s|^r \quad \text{for a.e } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

where $C_\epsilon = C_{\delta(\epsilon)} = \frac{\|a\|_\infty}{\delta^{r-1}} + \frac{\epsilon}{r}$.

We end this section by showing an estimate involving field potential G . First, extend G outside $\Omega \times \Omega$ and any function $u, v, w \in H_{0,L}^1(\mathcal{C})$ outside $\Omega \times \{0\}$ in a trivial way. By Hölder's inequality, if $2q \in [2, 2^\sharp]$, we get

$$\left| \int_\Omega \langle G, v^2 \rangle uw \, dx \right| \leq |\langle W, v^2 \rangle|_{q'} |uw|_q = |W * v^2|_{q'} |uw|_q,$$

where $*$ denotes the usual convolution product in \mathbb{R}^N and where we have denoted traces on $\Omega \times \{0\}$ simply by functions themselves. Now, apply Young's inequality for convolutions, choosing q so that $\frac{1}{q'} = \frac{1}{r} + \frac{1}{q} - 1$, that is $q = \frac{2r}{2r-1}$, so that from the previous inequality we get

$$(3.10) \quad \left| \int_\Omega \langle G, v^2 \rangle uw \, dx \right| \leq |W|_r |v|_{2q}^2 |uw|_q \leq |W|_r |v|_{2q}^2 |u|_{2q} |w|_{2q}.$$

We remark that, since $r \in (\frac{N}{2}, \infty)$, we have $1 < q < N/(N-1)$. Finally, by the interpolation and the Sobolev inequalities, we get that there exists $C_G > 0$ such that

$$(3.11) \quad \left| \int_\Omega \langle G, v^2 \rangle uw \, dx \right| \leq C_G \|v\|^2 \|u\| \|w\| \quad \text{for any } u, v, w \in H_{0,L}^1(\mathcal{C}).$$

4. REGULARITY OF WEAK SOLUTIONS

In this section we briefly complement Cabré–Tan's results on regularity of weak solutions: such results seem to be very natural, and are related to the regularity properties established in [4, Proposition 3.1] for $m = 0$ and in [6, Theorem 3.2 and Proposition 3.9].

Proposition 1. *Suppose $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded domain with $\alpha \in (0, 1)$. Then, under hypotheses **(H)** and **(G)**, all weak solutions v of problem (2.1) are of class $L^\infty(\mathcal{C}) \cap C^\alpha(\overline{\Omega})$ and $u \in L^\infty(\Omega)$, being $u(\cdot) = v(\cdot, 0)$ a solution of (1.6). Moreover, for every $p \in [1, \infty]$ there exists $M_p > 0$ such that*

$$\|v\|_{L^p(\mathcal{C})} \leq M_p$$

and also

$$\|u\|_{L^p(\Omega)} \leq M_p.$$

Proof. In order to prove the first statement, we need only minor changes in the proof of [6, Theorem 3.2], and for this here we will be sketchy. As usual, set

$v_T = \max\{v_+, T\}$ and, fixed $\beta > 0$, apply (2.3) with $w = vv_T^{2\beta} \in H_{0,L}^1(\mathcal{C})$. Then we get

$$\begin{aligned} \|vv_T^\beta\|^2 &= \int_{\mathcal{C}} \left(|D(vv_T^\beta)|^2 + (vv_T^\beta)^2 \right) dx dx_{N+1} \\ &\leq c_\beta \int_{\Omega} \left(\omega v^2 v_T^{2\beta} + \lambda \langle G, v^2 \rangle v^2 v_T^{2\beta} + f(x, v) vv_T^{2\beta} \right) dx. \end{aligned}$$

By **(G)** and **(Hi)** we can easily recover estimate (3.3) of [6], obtaining

$$\int_{\mathcal{C}} \left(|D(v_+^{\beta+1})|^2 + (v_+^{\beta+1})^2 \right) dx dx_{N+1} \leq c_\beta \int_{\Omega} \left(cv_+^{2\beta+2} dx + gv_+^{2\beta+2} \right) dx$$

for some $g \in L^N(\Omega)$. Hence, proceed as in [6] to obtain the claim. \square

Now, in order to state the regularity results, let us consider the problem

$$(4.1) \quad \begin{cases} \sqrt{-\Delta + m^2} u = g(x) & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and its related extended one

$$(4.2) \quad \begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathcal{C}, \\ -\frac{\partial v}{\partial x_{N+1}} = g(x) & \text{on } \Omega \times \{0\} \\ v = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times [0, \infty). \end{cases}$$

From now on, if v solves (4.2), the associated solution of (4.1) will be denoted by $u = \text{tr } v$, meaning that $u(\cdot) = v(\cdot, 0)$. Adapting the proof of [4, Proposition 3.1], and representing the space of all traces on $\Omega \times \{0\}$ of functions in $H_{0,L}^1(\mathcal{C})$ by the symbol \mathcal{V}_0 , we immediately have the following proposition:

Proposition 2. *Let $\alpha \in (0, 1)$, Ω be a $C^{2,\alpha}$ bounded domain of \mathbb{R}^N , $v \in H_{0,L}^1(\mathcal{C})$ be the weak solution of (4.2), $u = \text{tr } v$ be the weak solution of (4.1) and $g \in \mathcal{V}_0^* \cap L^p(\Omega)$ for some $p \in (1, \infty)$. Then $v \in W^{2,p}(\Omega \times (0, R))$ for all $R > 0$. If $g \in C^\alpha(\overline{\Omega} \times \mathbb{R})$ and $g|_{\partial\Omega} \equiv 0$, then $v \in C^{1,\alpha}(\overline{\mathcal{C}})$, $u \in C^{1,\alpha}(\overline{\Omega})$.*

5. CONSTANT SIGN SOLUTIONS

Our main result is the following

Theorem 1. *Under hypotheses **(H)** and **(G)**, for any $\lambda > 0$ problem (2.1) admits two non trivial bounded solutions, one strictly positive and one strictly negative in Ω .*

The proof of Theorem 1 is based on an application of the Mountain Pass Theorem to functionals $J_+(v)$, $J_-(v)$, defined in $H_{0,L}^1(\mathcal{C})$ as follows:

$$(5.1) \quad \begin{aligned} J_\pm(v) &= \frac{1}{2} \int_{\mathcal{C}} [|Dv|^2 + m^2 v^2] dx dx_{N+1} \\ &\quad - \int_{\Omega} \left[\frac{\omega}{2} v^2 + \frac{\lambda}{4} \langle G, v^{\pm 2} \rangle v^{\pm 2} + F_\pm(x, v) \right] dx. \end{aligned}$$

Here $F_\pm(x, v) = F(x, \pm v^\pm)$, where $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$ denote the positive and the negative part of v , respectively.

However, though verifying the geometrical assumptions of the mountain pass is not very hard, thanks to some inequalities established above, the verification of the compactness condition is the hardest part. Moreover, since our assumptions are

very general and do not imply a growth of order $q > 2$ at infinity, the usual Palais–Smale condition has to be replaced by the generally weaker Cerami condition:

Definition 2. *Let X be a Banach space with topological dual X^* . A C^1 functional $J : X \rightarrow \mathbb{R}$ is said to satisfy the Cerami condition - (C) for short - if every sequence $(u_n)_n \subset X$ such that $(J(u_n))_n$ is bounded and $(1 + \|u_n\|)J'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$, has a convergent subsequence.*

As shown by Bartolo-Benci-Fortunato [1], such a condition can successfully replace the Palais–Smale condition in proving a Deformation Theorem, and consequently a minimax theory for critical values. In particular, the classical Mountain Pass Theorem holds under this compactness condition and we will apply such a version (see [12, Corollary 5.2.7]).

Now, we will check the Mountain Pass hypothesis for J_+ , but analogous results hold true with minor changes for J_- (and J). In this way we will exhibit the existence of a positive and of a negative solution.

First, we show that J_+ has a strict local minimum at the origin: take $v \in H_{0,L}^1(\mathcal{C})$, then, by (3.11) and (3.9) we find

$$\begin{aligned}
 J_+(v) &\geq \frac{1}{2} \|Dv\|_2^2 + \frac{m^2}{2} \|v\|_2^2 - \frac{\omega}{2} |v|_2^2 - \lambda C_G \|v\|^4 \\
 (5.2) \quad &- \int_{\Omega} \frac{\theta(x) + \epsilon}{2} (v^+)^2 dx - C_{\epsilon} |v^+|_r^r \\
 &\geq \frac{1}{2} \left[\|Dv\|_2^2 + m^2 \|v\|_2^2 - (\omega + \theta_{\infty} + \epsilon) |v|_2^2 \right] - \lambda C_G \|v\|^4 - \tilde{C}_{\epsilon} \|v\|^r,
 \end{aligned}$$

for some $\tilde{C}_{\epsilon} > 0$.

If $\omega + \theta_{\infty} + \epsilon \leq 0$, the claim follows immediately, since $r > 2$. If $\omega + \theta_{\infty} + \epsilon > 0$, by **Hiv**) we can suppose that $\theta_{\infty} + \epsilon < m - \omega$, and using (3.6), we find a positive constant \tilde{c} such that

$$\begin{aligned}
 J_+(v) &\geq \frac{1}{2} \left(1 - \frac{\omega + \theta_{\infty} + \epsilon}{m} \right) \|Du\|_2^2 + \frac{1}{2} \left(m^2 - m(\omega + \theta_{\infty} + \epsilon) \right) \|v\|_2^2 \\
 (5.3) \quad &- C_G \|v\|^4 - C_{\epsilon} \|v\|^r \\
 &\geq \tilde{c} \|v\|^2 - C_G \|v\|^4 - C_{\epsilon} \|v\|^r.
 \end{aligned}$$

Thus, 0 is a strict local minimum for J_+ , and there exists $\rho > 0$ such that

$$(5.4) \quad 0 = J_+(0) < \inf\{J_+(u) : \|u\| = \rho\} := \eta_+.$$

Next, by (3.9), we have that for every $v \in H_{0,L}^1(\mathcal{C})$

$$\begin{aligned}
 J_+(v) &\leq \frac{\max\{1, m^2\}}{2} \|v\|^2 - \frac{\omega}{2} |v|_2^2 - \frac{\lambda}{4} \int_{\Omega} \langle G, (v^+)^2 \rangle v^{+2} dx \\
 (5.5) \quad &+ \frac{\theta_{\infty} + \epsilon}{2} |v|_2^2 + C_{\epsilon} |v|_r^r
 \end{aligned}$$

Thus, if $t > 0$, and we choose a nonnegative $v \in H_{0,L}^1(\mathcal{C})$, there exist positive constants c_1, c_2, c_3, c_4, c_5 such that

$$(5.6) \quad J_+(tv) \leq c_1 t^2 - c_2 t^2 - c_3 t^4 + c_4 t^2 + c_5 t^r \xrightarrow{t \rightarrow \infty} -\infty,$$

since $r < 4$ for all $N \geq 2$.

In this way we have proved the validity of the geometric conditions of the Mountain Pass Theorem. Next, we proceed by showing the compactness hypothesis in the form of the Cerami condition.

5.1. Verification of (C). Let $(u_n)_n$ be a Cerami sequence in $H_{0,L}^1(\mathcal{C})$, i.e. a sequence such that

$$(5.7) \quad \begin{cases} |J_+(u_n)| \leq M \quad \forall n \in \mathbb{N} \text{ and} \\ (1 + \|u_n\|) J'_+(u_n) \xrightarrow{n \rightarrow \infty} 0 \text{ in } [H_{0,L}^1(\mathcal{C})]^* \end{cases}$$

for some $M > 0$. Now, we prove that $(u_n)_n$ admits a converging subsequence.

Lemma 1. *The sequence $(u_n)_n$ is bounded.*

Proof. From (5.7) we have

$$(5.8) \quad |J'_+(u_n)h| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in H_{0,L}^1(\mathcal{C}),$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Writing (5.8) explicitly, using the analogue of (2.6) for J_+ , we find

$$(5.9) \quad \left| \int_{\mathcal{C}} [Du_n \cdot Dh + m^2 u_n h] dx dx_{N+1} - \omega \int_{\Omega} u_n h dx - \lambda \int_{\Omega} \langle G, (u_n^+)^2 \rangle u_n^+ h dx - \int_{\Omega} f_+(x, u_n) h dx \right| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|}$$

Now, in (5.9) we choose alternatively $h = u_n^-$ and $h = u_n^+$, so that we find, respectively,

$$(5.10) \quad \left| \|Du_n^-\|_2^2 + m^2 \|u_n^-\|_2^2 - \omega |u_n^-|_2^2 \right| \leq \frac{\epsilon_n \|u_n^-\|}{1 + \|u_n\|},$$

and

$$(5.11) \quad \left| \|Du_n^+\|_2^2 + m^2 \|u_n^+\|_2^2 - \omega |u_n^+|_2^2 - \lambda \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx - \int_{\Omega} f(x, u_n^+) u_n^+ dx \right| \leq \frac{\epsilon_n \|u_n^+\|}{1 + \|u_n\|}.$$

By using (3.6), from (5.10) we immediately see that $(u_n^-)_n$ is bounded in $H_{0,L}^1(\mathcal{C})$.

Now, we rewrite $J_+(u_n)$ as sum of two components, acting on u_n^+ and u_n^- separately:

$$(5.12) \quad \begin{aligned} J_+(u_n) &= \frac{1}{2} \left[\|Du_n^-\|_2^2 + m^2 \|u_n^-\|_2^2 - \omega |u_n^-|_2^2 \right] \\ &\quad + \frac{1}{2} \left[\|Du_n^+\|_2^2 + m^2 \|u_n^+\|_2^2 - \omega |u_n^+|_2^2 \right] \\ &\quad - \frac{\lambda}{4} \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx - \int_{\Omega} F(x, u_n^+) dx. \end{aligned}$$

By (5.12) we can write

$$\begin{aligned}
 J_+(u_n) &= \frac{1}{2} J'_+(u_n)(u_n^-) + \frac{1}{2} J'_+(u_n)(u_n^+) + \frac{\lambda}{4} \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx \\
 &\quad + \frac{1}{2} \int_{\Omega} f(x, u_n^+) u_n^+ dx - \int_{\Omega} F(x, u_n^+) dx \\
 (5.13) \quad &= \frac{1}{2} J'_+(u_n)(u_n^-) + \frac{1}{2} J'_+(u_n)(u_n^+) + \frac{\lambda}{4} \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx \\
 &\quad + \frac{1}{2} \int_{\Omega} \sigma(x, u_n^+) dx.
 \end{aligned}$$

The first two terms of the last side are bounded by ϵ_n , see (5.8). Then,

$$J_+(u_n) \geq -\epsilon_n + \frac{\lambda}{4} \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx + \frac{1}{2} \int_{\Omega} \sigma(x, u_n^+) dx.$$

In addition, the last term is limited from below, since condition **Hiii**) implies that

$$0 = \sigma(x, 0) \leq \sigma(x, t) + \beta^*(x) \quad \forall t \geq 0.$$

As a consequence, from (5.13) we get

$$J_+(u_n) \geq -\epsilon_n + \frac{\lambda}{4} \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx - \frac{1}{2} |\beta^*|_1$$

Finally, from the bound on $J_+(u_n)$, see (5.7, and the non negativity of G , we get that there exists $M \geq 0$ such that

$$(5.14) \quad \int_{\Omega} \langle G, (u_n^+)^2 \rangle (u_n^+)^2 dx \in [0, M] \quad \forall n \in \mathbb{N}.$$

Let us now remark that, by (3.6),

$$\|D \cdot\|_2^2 + m^2 \|\cdot\|_2^2 - \omega \|\cdot\|_2^2$$

defines a quantity which is equivalent to $\|\cdot\|^2$ in $H_{0,L}^1(\mathcal{C})$, which we shall denote by $\|\cdot\|^2$, from now on. Using this notation, starting from (5.11), by using (5.14), we get the existence of $M_2 \geq 0$ such that

$$(5.15) \quad \left| \|\cdot\|^2 u_n^+ - \int_{\Omega} f(x, u_n^+) u_n^+ \right| \leq M_2 \text{ for all } n \in \mathbb{N}.$$

Moreover, from the bound on $J_+(u_n)$ given by (5.7), from the bound on the Green-potential term G in (5.14) and from the bound on u_n^- , starting from (5.12), we also get

$$(5.16) \quad \left| \frac{1}{2} \|\cdot\|^2 u_n^+ - \int_{\Omega} F(x, u_n^+) \right| \leq M_3 \text{ for all } n \in \mathbb{N}$$

for some $M_3 \geq 0$. Combining both (5.15) and (5.16) we get

$$(5.17) \quad \left| \int_{\Omega} \sigma(x, u_n^+) \right| \leq M_4 \text{ for all } n \in \mathbb{N}$$

for some $M_4 \geq 0$.

Claim: u_n^+ is bounded. We prove it by contradiction. Suppose that $(u_n^+)_n$ is not bounded; then, we may assume that $\|u_n^+\| \xrightarrow{n} \infty$, and that

$$y_n := \frac{u_n^+}{\|u_n^+\|} \xrightarrow{n} y \text{ in } H_{0,L}^1(\mathcal{C}),$$

and by (3.2), we can also assume that

$$(5.18) \quad \begin{cases} y_n \xrightarrow{n} y \text{ in } L^q(\Omega) \text{ for every } q \in \left[1, \frac{2N}{N-1}\right) \\ y_n(x) \xrightarrow{n} y(x) \geq 0 \text{ for a.e. } x \text{ in } \Omega. \end{cases}$$

We distinguish two cases, according to whether $y \not\equiv 0$ or $y \equiv 0$. In the former case we consider the set $Z = \{x \in \Omega : y(x) = 0\}$, whose complementary set Z^c has positive measure. It is clear that

$$u_n^+ \xrightarrow{n} \infty \text{ a.e. in } Z^c.$$

Therefore, by **Hii**), we get

$$(5.19) \quad \frac{F(x, u_n^+)}{\|u_n^+\|^2} = \frac{F(x, u_n^+)}{|u_n^+|^2} \frac{|u_n^+|^2}{\|u_n^+\|^2} \xrightarrow{n} \infty \text{ a.e. in } Z^c.$$

Now, combining **Hi**) and **Hii**), we see that there exists $g \in L^1(\Omega)$ such that $\frac{F(x, u_n^+)}{\|u_n^+\|^2} \geq g(x)$ for a.e. $x \in \Omega$, and we can use Fatou's lemma to obtain

$$(5.20) \quad \begin{aligned} \liminf \int_{\Omega} \frac{F(x, u_n^+)}{\|u_n^+\|^2} &\geq \int_{Z^c} \liminf \frac{F(x, u_n^+)}{\|u_n^+\|^2} + \int_Z \liminf \frac{F(x, u_n^+)}{\|u_n^+\|^2} \\ &\geq C + \int_{Z^c} \liminf \frac{F(x, u_n^+)}{\|u_n^+\|^2} = \infty, \end{aligned}$$

where C is a constant.

On the other hand, (5.16) implies that

$$\left| 1 - \frac{F(x, u_n^+)}{\|u_n^+\|^2} \right| \leq \frac{M_3}{\|u_n^+\|^2},$$

in contradiction with (5.20). Hence, in this case the claim is proved.

We now turn to the second case, i.e. $y \equiv 0$ in $H_{0,L}^1(\mathcal{C})$. Let us set $\gamma_n(t) := J(tu_n^+)$, for $t \in [0, 1]$. The sequence $t_n = \operatorname{argmax}_{t \in [0,1]} \gamma_n(t)$ is well defined, since $\gamma_n \in C[0, 1]$.

For every $k \in (0, \|u_n^+\|)$ we set $\tilde{t}_n = \frac{k}{\|u_n^+\|}$, so that $\gamma_n(\tilde{t}_n) = J(ky_n)$ and $\tilde{t}_n \in (0, 1)$; thus

$$(5.21) \quad \begin{aligned} J(t_n u_n^+) &\geq J(\tilde{t}_n u_n^+) = \frac{1}{2} \|\tilde{t}_n u_n^+\|^2 - \frac{\lambda}{4} \int_{\Omega} \langle G, (\tilde{t}_n u_n^+)^2 \rangle |\tilde{t}_n u_n^+|^2 - \int_{\Omega} F(x, \tilde{t}_n u_n^+) \\ &= \frac{1}{2} k^2 - \frac{\lambda}{4} \int_{\Omega} \langle G, (\tilde{t}_n u_n^+)^2 \rangle |\tilde{t}_n u_n^+|^2 - \int_{\Omega} F(x, \tilde{t}_n u_n^+). \end{aligned}$$

From (5.18) and **Hi**), we see that

$$\int_{\Omega} F(x, \tilde{t}_n u_n^+) \xrightarrow{n \rightarrow \infty} 0.$$

In addition, by the Lebesgue Theorem and (3.11), we have that

$$\int_{\Omega} \langle G, (\tilde{t}_n u_n^+)^2 \rangle |\tilde{t}_n u_n^+|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Hence, from (5.21), given $M > 0$, there exists $N = N(M)$ such that

$$(5.22) \quad \begin{aligned} \left| \int_{\Omega} F(x, \tilde{t}_n u_n^+) \right| &\leq \frac{M}{8}, \\ \lambda \int_{\Omega} \langle G, (\tilde{t}_n u_n^+)^2 \rangle |\tilde{t}_n u_n^+|^2 &\leq \frac{M}{2} \end{aligned}$$

for all $n \geq N$. Choosing $k = \sqrt{M}$, from (5.21) and (5.22), we finally get

$$J(t_n u_n^+) \geq \frac{M}{2} \text{ for every } n > N,$$

that is:

$$(5.23) \quad J(t_n u_n^+) \xrightarrow{n \rightarrow \infty} \infty.$$

The limit in (5.23) implies that $t_n \neq 0$ for n large enough. On the other hand, we also have $t_n \neq 1$. Indeed, if $t_n = 1$, we would have

$$(5.24) \quad J(u_n^+) = \frac{1}{2} \|u_n^+\|^2 - \frac{\lambda}{4} \int_{\Omega} \langle G, (u_n^+)^2 \rangle |u_n^+|^2 - \int_{\Omega} F(x, u_n^+),$$

which is bounded, thanks to (5.14) and (5.16). Eventually, we conclude that $t_n \in (0, 1)$ for n large enough; this implies that

$$(5.25) \quad \begin{aligned} 0 &= t_n \frac{d}{dt} J(tu_n^+) \Big|_{t=t_n} = t_n \langle J'(t_n u_n^+), u_n^+ \rangle = \langle J'(t_n u_n^+), t_n u_n^+ \rangle \\ &= \|t_n u_n^+\|^2 - \lambda t_n^4 \int_{\Omega} \langle G, (u_n^+)^2 \rangle |u_n^+|^2 - \int_{\Omega} f(x, t_n u_n^+) t_n u_n^+ \\ &= \|t_n u_n^+\|^2 - \lambda t_n^4 \int_{\Omega} \langle G, (u_n^+)^2 \rangle |u_n^+|^2 - 2 \int_{\Omega} F(x, t_n u_n^+) - \int_{\Omega} \sigma(x, t_n u_n^+) \end{aligned}$$

Using hypothesis **(Hiii)**, from (5.25) and (5.17), we get the existence of a positive constant M_5 such that

$$(5.26) \quad \begin{aligned} \|t_n u_n^+\|^2 - \lambda t_n^4 \int_{\Omega} \langle G, (u_n^+)^2 \rangle |u_n^+|^2 - 2 \int_{\Omega} F(x, t_n u_n^+) \\ = \int_{\Omega} \sigma(x, t_n u_n^+) \leq \int_{\Omega} \sigma(x, u_n^+) + |\beta^*|_1 \leq M_5, \end{aligned}$$

for every n large enough.

Finally, we show that (5.23) implies that the left-hand-side of (5.26) diverges, thus obtaining a contradiction. Indeed,

$$2J(t_n u_n^+) = \|t_n u_n^+\|^2 - \frac{\lambda}{2} t_n^4 \int_{\Omega} \langle G, (u_n^+)^2 \rangle |u_n^+|^2 - 2 \int_{\Omega} F(x, \tilde{t}_n u_n^+) \xrightarrow{n \rightarrow \infty} \infty;$$

but, by (5.14), we obtain the announced contradiction.

As a consequence, $(u_n^+)_n$ is bounded. From (5.10), we see that the whole sequence $(u_n)_n$ is bounded in $H_{0,L}^1(\mathcal{C})$, as claimed. \square

Lemma 2. $(u_n)_n$ converges strongly in $H_{0,L}^1(\mathcal{C})$.

Proof. First, being $(u_n)_n$ bounded in $H_{0,L}^1(\mathcal{C})$, up to subsequences, we may assume that there exists $u \in H_{0,L}^1(\mathcal{C})$ such that

$$(5.27) \quad \begin{cases} u_n \rightharpoonup_n u \text{ in } H_{0,L}^1(\mathcal{C}), \\ u_n \xrightarrow[n]{L^q(\Omega)} u \text{ for every } q \in \left[1, \frac{2N}{N-1}\right) \\ u_n \xrightarrow[n]{\text{a.e.}} u \text{ in } \Omega. \end{cases}$$

We claim that $(u_n)_n$ converges strongly to u in $H_{0,L}^1(\mathcal{C})$. In order to prove this claim we will exploit (5.7), and in particular the fact that

$$J'_+(u_n) \xrightarrow[n \rightarrow \infty]{} 0 \text{ in } (H_{0,L}^1(\mathcal{C}))^*,$$

which implies that

$$(5.28) \quad J'_+(u_n)(u_n - u) \xrightarrow[n \rightarrow \infty]{} 0.$$

But

$$(5.29) \quad J'_+(u_n)(u_n - u) = \|u_n\|^2 - \int_{\mathcal{C}} Du_n \cdot Du \, dx dx_{N+1}$$

$$(5.30) \quad -\lambda \int_{\Omega} \langle G, (u_n^+)^2 \rangle u_n^+(u_n - u) dx - \int_{\Omega} f(x, u_n^+)(u_n - u) dx.$$

Then, showing that (5.30) goes to 0 as $n \rightarrow \infty$, (5.28) and (5.29) imply that $u_n \rightarrow u$ in the Hilbert space $H_{0,L}^1(\mathcal{C})$. First, the convergence

$$\int_{\Omega} \langle G, (u_n^+)^2 \rangle u_n^+(u_n - u) dx \xrightarrow[n \rightarrow \infty]{} 0$$

follows directly from 3.10. Then, from **Hi**),

$$(5.31) \quad \left| \int_{\Omega} f(x, u_n^+)(u_n - u) dx \right| \leq \int_{\Omega} a(x)|u_n - u| dx + c \int_{\Omega} |u_n^+|^{r-1}|u_n - u| dx,$$

and from (5.27) we have that all integrals in (5.31) go to 0 as $n \rightarrow \infty$.

We have thus proved that J_+ satisfies the Cerami condition. \square

Proof of Theorem 1. We apply the Mountain Pass Theorem obtaining the existence of a critical point $u_0 \in H_{0,L}^1(\mathcal{C})$ for J_+ , with $u_0 \neq 0$, $J_+(u_0) > 0$ and

$$(5.32) \quad J'_+(u_0) = 0$$

Applying (5.32) to u_0^- we see that

$$(5.33) \quad J'_+(u_0)(u_0^-) = \|u_0^-\|^2 = 0$$

so that $u_0 \geq 0$, $J'(u_0) = J'_+(u_0) = 0$ and, consequently, u_0 is a weak nonnegative and non trivial solution of problem (2.1). Furthermore, the maximum principle implies that $u_0 > 0$ in Ω , see [20, Proposition 3.2].

In the same way, using the functional J_- , it is possible to obtain a solution $v_0 < 0$ in Ω .

Finally, by Proposition 1, we get the bound on the solutions. \square

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REFERENCES

1. BARTOLO P., BENCI V. AND FORTUNATO D., *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Anal. **7**, 981–1012 (1983).
2. BENCI V. AND FORTUNATO D., *An eigenvalue problem for the Schrödinger–Maxwell equations*, Topolog. Meth. Nonlin. Analysis **11**, 283–293 (1998).
3. BENCI V. AND FORTUNATO D., *Spinning Q-Balls for the Klein-Gordon-Maxwell Equations*, Commun. Math. Phys. **295**, 639–668 (2010).
4. CABRÉ X. AND TAN J., *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224**, 2052–2093 (2010).
5. CAFFARELLI L. AND SILVESTRE L., *An Extension Problem Related to the Fractional Laplacian*, Comm. Partial Differential Equations **32**, 1245–1260 (2007).
6. COTI ZELATI V. AND NOLASCO M., *Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations*, Rend. Lincei Mat. Appl. **2**, 51–72 (2011).
7. D’APRILE T. AND MUGNAI D., *Non-Existence Results for the Coupled Klein-Gordon-Maxwell Equations*, Adv. Nonlinear Stud. **4**, 307–322 (2004).
8. D’APRILE T. AND MUGNAI D., *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger–Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134**, 893–906 (2004).
9. A. ELGART AND B. SCHLEIN, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math. **60**, 500545 (2007).
10. FRÖHLICH J., JONSSON B.L.G. AND LENZMANN E., *Boson Stars as Solitary Waves*, Comm. Math. Phys. **274**, 1–30 (2007).
11. FRÖHLICH J. AND LENZMANN E., *Blow-Up for Nonlinear Wave Equations describing Boson Stars*, Comm. Pure Appl. Math. **60**, 1691–1705 (2007).
12. GASINSKI L. AND PAPAGEORGIOU N.S., *Nonlinear analysis*. Ser. Math. Anal. Appl. **9**, Chapman & Hall/CRC, Boca Raton, FL, 2006.
13. GÜNTHER M. AND WIDMAN K.-O., *The Green function for uniformly elliptic equations*, Manuscripta Math. **37**, 303–342 (1982).
14. LI Y., WANG Z.-Q. AND ZENG J., *Ground states of nonlinear Schrödinger equations with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23**, 829–837 (2006).
15. LI G. AND YANG C., *The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p -Laplacian type without the Ambrosetti-Rabinowitz condition*, Nonlinear Anal. **72**, 4602–4613, (2010).
16. LIEB E.H. AND THIRRING W.E., *Gravitational collapse in quantum mechanics with relativistic kinetic energy*, Ann. Physics **155**, 494–512 (1984).
17. E.H. LIEB AND H.-T. YAU, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112**, 147174 (1987).
18. LIU S., *On superlinear problems without the Ambrosetti and Rabinowitz condition*, Nonlinear Anal. **73**, 788–795, (2010).
19. LIU Z. AND WANG Z.-Q., *On the Ambrosetti-Rabinowitz superlinear condition*, Adv. Nonlinear Stud. **4**, 561–572 (2004).
20. MARINELLI A. AND MUGNAI D., *The generalized logistic equation with indefinite weight driven by the square root of the Laplacian*, Nonlinearity **27**, 2361–2376 (2014).
21. MIYAGAKI O. AND SOUTO M.A.S., *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, J. Differential Equations **245**, 3628–3638, (2008).
22. MUGNAI D., *Coupled Klein-Gordon and Born-Infeld type equations: looking for solitary waves*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **460**, 1519–1528 (2004).
23. MUGNAI D., *Solitary waves in Abelian Gauge Theories with strongly nonlinear potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27**, 1055–1071 (2010).
24. MUGNAI D., *The pseudorelativistic Hartree equation with a general nonlinearity: existence, non existence and variational identities*, Adv. Nonlinear Stud. **13**, 799–823 (2013).
25. MUGNAI D., *The Schrödinger–Poisson System with Positive Potential*, Comm. Partial Differential Equations **36**, 1099–1117 (2011).
26. MUGNAI D. AND PAPAGEORGIOU N. S., *Wang’s multiplicity result for superlinear (p,q) -equations without the Ambrosetti-Rabinowitz condition*, Trans. Amer. Math. Soc. **366**, 4919–4937 (2014).

- 27. MUGNAI D. AND RINALDI M., *Spinning Q-balls in Abelian Gauge Theories with positive potentials: existence and non existence*, Calc. Var. Partial Differential Equations **53**, 1–27 (2015).
- 28. SICKEL W. AND SKRZYPCZAK L., *Radial subspaces of Besov and Lizorkin-Triebel spaces: extended Strauss lemma and compactness of embeddings*, J. Fourier Anal. Appl. **6**, 639–662 (2000).

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